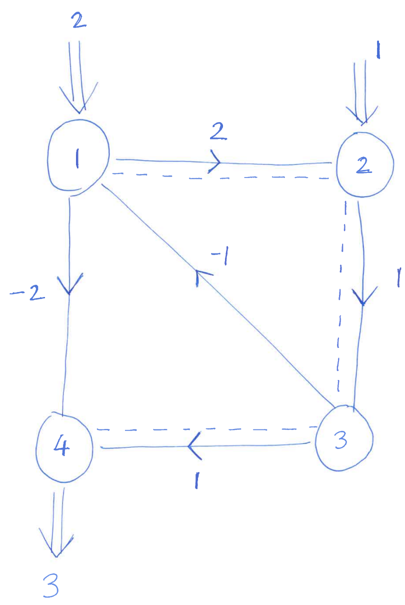
Backpaper - Optimization (2019-20)
Time: 3 hours.
Attempt all questions. There are a total of 55 points, the maximum you can score is 50.

1. Consider the network flow problem on the directed graph shown below. The numbers next to each directed arc $\to$ is the *cost* associated to the arc, while the numbers next to $\Rightarrow$ is the external *supply/demand* at the node.



Denote by $\mathbf{c}$ the vector of costs corresponding to the arc. We are interested in minimizing the total cost $\mathbf{c}^T\mathbf{f}$, where the flow vector $\mathbf{f}$ satisfies the flow conservation equations and $\mathbf{f} \geq \mathbf{0}$.

   (a) Find an optimal basic feasible solution (feasible tree solution) to the problem. **[7 points]**

   (b) Find the optimal cost for the problem. **[3 points]**

   Note: If you are implementing the simplex algorithm, you can find an initial feasible basic/tree solution by considering the tree indicated by dashed lines.

2. Write the following problem in *standard form*: minimize $-2x_1 - x_2$

$$\text{subject to} \qquad x_1 - x_2 \leq 2, \quad x_1 + x_2 \leq 6, \quad x_1 \geq 0,\ x_2 \in \mathbf{R}.$$ **[5 points]**

3. Minimize $-2x_1 + x_2 - 2x_3$

$$\begin{aligned} \text{subject to} \qquad 2x_1 + x_2 + x_4 &= 10, \\ x_1 + 2x_2 - 2x_3 + x_5 &= 20, \\ x_2 + 2x_3 + x_6 &= 5, \\ x_1, x_2, \cdots, x_6 &\geq 0. \end{aligned}$$ **[5 points]**

4. Let $\mathbf{B} = \{\mathbf{x}_1, \cdots \mathbf{x}_n\}$ be an orthonormal basis of an inner product space V. Then show for any $\mathbf{x} \in V$

$$\mathbf{x} = \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{x}_j \rangle \, \mathbf{x}_j,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on V. **[5 points]**

5. Solve the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ (first check whether it is consistent, and then find the general solution if the system is consistent). Also find for $\mathbf{A}$, a g-inverse, the rank, a rank factorisation and a basis of the null space.

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 1 \\ 4 & 0 & 4 \\ 3 & 1 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$$ **[5 points]**

6. Define a *basic feasible solution* (bfs) of a polyhedron P defined by the inequality constraints $\mathbf{a}_i^T \mathbf{x} \geq b_i$, $i = 1, 2 \cdots, m_1$ and the equality constraints $\mathbf{a}_i^T \mathbf{x} = b_i$, $i = m_1 + 1, \cdots, m_1 + m_2$. **[4 points]**

7. Suppose that the polyhedron $P = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{a}_i^T \mathbf{x} \geq b_i, \, i = 1, 2 \cdots, m\}$ is nonempty. Then show that if there are n vectors among $\mathbf{a}_1, \cdots, \mathbf{a}_m$ that are linearly independent, then the polyhedron P does not contain a line. **[5 points]**

8. Write down the dual of the following problem: minimize $x_1 - x_2$

$$\begin{aligned} \text{subject to} \qquad 2x_1 + 3x_2 - x_3 + x_4 &\leq 0, \\ 3x_1 + x_2 + 4x_3 - 2x_4 &\geq 3, \\ -x_1 - x_2 + 2x_3 + x_4 &= 6, \\ x_1 &\leq 0, \\ x_2, \, x_3 &\geq 0. \end{aligned}$$ **[4 points]**

9. For each of the statements below, state whether it is true or false. If true, prove it and if false, give a counterexample.

   (a) Consider the problem of minimizing $\mathbf{c}^T \mathbf{x}$ over a polyhedron P. If the polyhedron has an extreme point then the minimization problem has a finite optimal value. **[4 points]**

   (b) Consider the problem of minimizing $\mathbf{c}^T \mathbf{x}$ over a polyhedron P. If there is more than one optimal solution, then there are uncountably many optimal solutions. **[4 points]**

   (c) Consider the dual problem to the primal problem: minimize $\mathbf{c}^T \mathbf{x}$ over a polyhedron P. The dual problem is feasible. **[4 points]**